

# Inverse resonance scattering for Jacobi operators

Evgeny Korotyaev \*

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## Abstract

We consider the Jacobi operator  $(Jf)_n = a_{n-1}f_{n-1} + a_nf_{n+1} + b_nf_n$  on  $\mathbb{Z}$  with a real compactly supported sequences  $(a_n - 1)_{n \in \mathbb{Z}}$  and  $(b_n)_{n \in \mathbb{Z}}$ . We give the solution of two inverse problems (including characterization):  $(a, b) \rightarrow \{\text{zeros of the reflection coefficient}\}$  and  $(a, b) \rightarrow \{\text{bound states and resonances}\}$ . We describe the set of "iso-resonance operators  $J$ ", i.e., all operators  $J$  with the same resonances and bound states.

**Keywords:** Jacobi operator, resonances, inverse problem

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## 1 Introduction

We consider the Jacobi operator  $J$  acting on  $\ell^2$  and given by

$$(Jf)_n = a_{n-1}f_{n-1} + a_nf_{n+1} + b_nf_n, \quad n \in \mathbb{Z}, \quad f = (f_n)_{n \in \mathbb{Z}} \in \ell^2(\mathbb{Z}),$$

where a real compactly supported sequence  $q = (q_n)_{n \in \mathbb{Z}}$ ,  $q_{2n-1} = b_n$ ,  $q_{2n} = 1 - a_n$  satisfies

$$q = (q_n)_{n \in \mathbb{Z}} \in \mathfrak{X}_\nu^\tau = \mathfrak{X}_\nu^\tau(p) = \ell_{1+\nu, 2p-\tau}, \quad \text{for some } \nu, \tau \in \{0, 1\}, \quad p \in \mathbb{N},$$

$$\ell_{m,k} = \left\{ (q_n)_{n \in \mathbb{Z}} \in \ell^2(\mathbb{Z}) : q_m \neq 0, q_k \neq 0, q_{2s} < 1, q_n = 0, \text{ all } s \in \mathbb{Z}, n \in \mathbb{Z} \setminus [m, k] \right\}, \quad (1.1)$$

for some  $m, k \in \mathbb{Z}$ . For fixed  $p \geq 1$  the sequence  $q \in \mathfrak{X}_0^0$  has the max support and  $q \in \mathfrak{X}_1^1$  has the min support and  $q \neq 0$ . It is well known that the spectrum of  $J$  has the form

$$\sigma(J) = \sigma_{ac}(J) \cup \sigma_d(J), \quad \sigma_{ac}(J) = [-2, 2], \quad \sigma_d(J) \subset \mathbb{R} \setminus [-2, 2]. \quad (1.2)$$

Define the new variable  $z \in \mathbb{D}_1 = \{z \in \mathbb{C} : |z| < 1\}$  by  $\lambda = \lambda(z) = z + \frac{1}{z}$ . Here  $\lambda(z)$  is a conformal mapping from  $\mathbb{D}_1$  onto  $\mathbb{C} \setminus [-2, 2]$ . Denote by  $\psi^\pm = (\psi_n^\pm(z))_{n \in \mathbb{Z}}$  the fundamental solutions to

$$a_{n-1}\psi_{n-1}^\pm + a_n\psi_{n+1}^\pm + b_n\psi_n^\pm = (z + \frac{1}{z})\psi_n^\pm, \quad n \in \mathbb{Z}, \quad (1.3)$$

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\*School of Math., Cardiff Univ., Senghennydd Road, Cardiff, CF24 4AG, UK, e-mail: KorotyaevE@cf.ac.uk

$$\psi_n^+(z) = z^n, \quad n > p, \quad \text{and} \quad \psi_n^-(z) = z^{-n}, \quad n \leq 0, \quad |z| \leq 1. \quad (1.4)$$

Define the Wronskian  $\{f, u\}_n = a_n(f_n u_{n+1} - u_n f_{n+1})$  for sequences  $u = (u_n)_{-\infty}^\infty, f = (f_n)_{-\infty}^\infty$ . If  $f, u$  are some solutions of (1.3), then  $\{f, u\}_n$  does not depend on  $n$ . The following identities hold true:

$$\psi^+ = A\tilde{\psi}^- + B\psi^-, \quad \text{on } \mathbb{S}_0^1 = \mathbb{S}^1 \setminus \{\pm 1\}, \quad \tilde{\psi}^\pm = \psi^\pm(z^{-1}), \quad (1.5)$$

where  $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ , and  $\tilde{v} = v(z^{-1})$  for a function  $v(z)$  and

$$A = \frac{w}{1 - z^2}, \quad w = z\{\psi^+, \psi^-\}_n, \quad B = \frac{z^2 s}{\eta}, \quad s = \frac{\{\psi^+, \tilde{\psi}^-\}_n}{z^2}, \quad \eta = z - \frac{1}{z}. \quad (1.6)$$

Note that if  $q = 0$ , then  $w = 1 - z^2, A = 1, B = 0$ . The S-matrix is given by

$$\mathcal{S}_M(z) = \begin{pmatrix} A(z)^{-1} & R_-(z) \\ R_+(z) & A(z)^{-1} \end{pmatrix}, \quad z \in \mathbb{S}^1, \quad R_- = \frac{B}{A} = -\frac{z^3 s}{w}, \quad R_+ = -\frac{\tilde{B}}{A} = \frac{\tilde{s}}{zw}, \quad (1.7)$$

where  $\frac{1}{A}$  is the transmission coefficient and  $R_\pm$  is the reflection coefficient. For each  $z \in \mathbb{S}_0^1$  the scattering matrix  $\mathcal{S}_M$  is unitary and satisfies

$$|A(z)|^2 = 1 + |B(z)|^2 \Leftrightarrow w(z)w(z^{-1}) + \eta^2(z) = s(z)s(z^{-1}), \quad (1.8)$$

$$\det \mathcal{S}_M(z) = \frac{\overline{A}(z)}{A(z)} = -z^2 \frac{\overline{w}(z)}{w(z)} = -z^2 \frac{w(z^{-1})}{w(z)}. \quad (1.9)$$

In the case of compactly supported  $q$  it is convenient to work with the polynomials  $w, s$ . Thus  $\mathcal{S}_M$  is meromorphic in the complex plane  $\mathbb{C}$  and the poles of  $\mathcal{S}_M$  are the zeros of  $w$ . Let  $\lambda_j, j \in \mathbb{Z}_N = \{n_-, \dots, -1, 1, \dots, n_+\}, N = n_+ - n_- \geq 0$  be the bound states of  $J$  and  $\lambda_j = z_j + z_j^{-1}$ , where  $z_j$  are all zeros of  $w$  in  $\mathbb{D}_1$  and the sequence  $E_N = (z_j)_{j \in \mathbb{Z}_N}$  satisfies

$$-1 < z_{n_-} < \dots < z_{-1} < 0 < z_1 < \dots < z_{n_+} < 1 \quad \text{for some } \pm n_\pm \geq 0. \quad (1.10)$$

Recall that  $w$  has only these real zeros in  $\mathbb{D}_1$  (otherwise  $J$  has a non-real eigenvalue).

The main goal of this paper is to prove the following results:

- i) the mapping  $(a, b) \rightarrow \{\text{bound states, zeros of the reflection coefficient}\}$  is 1-to-1 and onto,
- ii) the mapping  $(a, b) \rightarrow \{\text{bound states, resonances and some sequence } \sigma = (\sigma_n)_0^m, \sigma_n = \pm 1, \}$  is 1-to-1 and onto,
- iii) we characterize "iso-resonance Jacobi operators  $J$ ", i.e., all operators  $J$  with the same resonances and bound states.

The inverse spectral problem consists of the following parts:

- i) Uniqueness. Prove that the spectral data uniquely determine the potential.
- ii) Characterization. Give conditions for some data to be the spectral data of some potential.
- iii) Reconstruction. Give an algorithm for recovering the potential from the spectral data.
- iv) A priori estimates. Obtain estimates of the potential in terms of the spectral data.

We define the class of scattering data  $(s, E_N)$ , where  $s(z)$  is a polynomial associated with the zeros of the reflection coefficient and  $E_N = (z_j)_{j \in \mathbb{Z}_N}$  is a sequence of all zeros of  $w$  in  $\mathbb{D}_1$ .

**Definition S.** By  $\mathcal{S}_\nu^\tau = \mathcal{S}_\nu^\tau(p), \nu, \tau \in \{0, 1\}, p \in \mathbb{N}$  we will denote the class of  $(f, K_f^N)$ , for some  $N \geq 0$ , where  $f$  is a real polynomial (i.e., a polynomial with real coefficients) given by

$$f = Cz^\nu \prod_1^m (z - \zeta_n), \quad \text{where } C \in \mathbb{R} \setminus \{0\}, \quad \zeta_n \in \mathbb{C} \setminus \{0\}, \quad m = 2p - 1 - \tau - \nu,$$

$K_f^N = (z_j)_{j \in \mathbb{Z}_N}$  is a sequence of zeros of the function  $f(z)f(z^{-1}) - \eta^2(z)$ , which satisfy

$$-1 < z_{n_-} < \dots < z_{-1} < 0 < z_1 < \dots < z_{n_+} < 1, \quad \pm n_\pm \geq 0, \quad N = n_+ - n_-, \quad (1.11)$$

$$(-1)^j z_j f(z_j^{\pm 1}) > 0, \quad j \in \mathbb{Z}_N, \quad \text{and} \quad (-1)^{n_\pm} f(\pm 1) \geq 0. \quad (1.12)$$

**Remark.** i) Below we will show that if  $q \in \mathfrak{X}_\nu^\tau$ , then  $(s, E_N) \in \mathcal{S}_\nu^\tau$ , where  $E_N = K_s^N$  is a sequence of all zeros of  $w$  in  $\mathbb{D}_1$ . Note that  $K_s^0 = \emptyset$ . ii) It is possible that the function  $s(z)s(z^{-1}) - \eta^2(z)$  has  $N_1 > N$  zeros on  $(-1, 1)$ , and it is important that we take some of them with the needed properties (1.12). In particular, we emphasize that  $E_N$  is not uniquely determined by  $s$ .

**Theorem 1.1.** Let  $\tau, \nu \in \{0, 1\}$  and  $m \geq 3$ . The mapping  $\mathfrak{J} : \mathfrak{X}_\nu^\tau \rightarrow \mathcal{S}_\nu^\tau$  given by  $\mathfrak{J}(q) = (s, E_N)$  is one-to-one and onto.

**Remark.** 1) In the proof of Theorem 1.1 we present algorithm for recovering the potential  $q$  from the spectral data  $(s, E_N)$ . The potential  $q$  is uniquely determined by the Marchenko equations (2.12)-(2.16) (in terms of  $(s, E_N)$ ). Here we use the results about the Marchenko equation from [Te]. Standard spectral data for the inverse problem for the Jacobi operator  $J$  are  $(s, z_j, m_j^\pm, j \in \mathbb{Z}_N)$ , where  $m_j^\pm$  is so-called norming constant given by

$$m_j^\pm = \sum_{n \in \mathbb{Z}} \psi_n^\pm(z_j)^2, \quad j \in \mathbb{Z}_N, \quad (1.13)$$

see [Te]. In Theorem 1.1 instead of the norming constants we need the condition (1.12).

2) In Section 3 we show simple examples of the scattering data for the case  $p = 1, 2$ .

3) We briefly indicate how to prove Theorem 1.1. Firstly, we show that if  $q \in \mathfrak{X}_\nu^\tau$ , then  $(s, E_N) \in \mathcal{S}_\nu^\tau$ . Here we check condition (1.12). Secondly, we consider the inverse mapping. Suppose  $(s, K_s^N) \in \mathcal{S}_\nu^\tau$ . Then in order to determine  $w$  we solve the functional equation (1.8) in some class of polynomials (see Theorem 3.2) and this gives the reflection coefficient  $R_\pm$ . Thirdly, (1.17), (1.18) yield the norming constants  $m_j^\pm$ . Then we check that  $R_\pm, z_j, m_j^\pm, j \in \mathbb{Z}_N$  satisfy conditions from Theorem 2.3 of Teschl [Te], which we recall for the sake of the reader in Sect.3. Then we obtain a bijection of our mapping.

4) Assume that  $s(\cdot) = 0$  for some  $q \in \mathfrak{X}_\nu^\tau$ . Then (1.8) yields  $w(z)w(z^{-1}) = -\tau(z)^2, z \in \mathbb{C} \setminus \{0\}$ . It is impossible since  $w$  is a polynomial and we have a contradiction. In fact we deduce that if  $s(\cdot) = 0$  for some compactly supported "potential"  $q$ , then  $q = 0$ .

Let  $\#(f, I)$  denote the number of zeros of a function  $f$  on the set  $I$ .

**Definition W.** By  $\mathcal{W}_\nu^\tau = \mathcal{W}_\nu^\tau(p), \nu, \tau \in \{0, 1\}, p \in \mathbb{N}$  we will denote the class of polynomials

$$w = C_w \prod_1^m (z - \rho_n), \quad C_w \in \mathbb{R} \setminus \{0\}, \quad m = 2p - 1 - \tau - \nu, \quad \rho_n \in \mathbb{C}, \quad w(0) > 0, \quad (1.14)$$

i)  $w$  is real on  $\mathbb{R}$  and  $|w(z)| \geq |\eta(z)|$  for any  $|z| = 1$ , where  $\eta = z - \frac{1}{z}$ ,  
ii)  $w$  has only real simple zeros  $z_j, j \in \mathbb{Z}_N$  in  $\overline{\mathbb{D}}_1$ , and the sequence  $E_N = (z_j)_{j \in \mathbb{Z}_N}$  and the function  $F(z) = w(z)w(z^{-1}) + \eta^2(z)$  satisfies

$$\begin{aligned} -1 < z_{n_-} < \dots < z_{-1} < 0 < z_1 < \dots < z_{n_+} < 1, \quad \text{for some } \pm n_{\pm} \geq 0, \quad N = n_+ - n_- \geq 0, \\ \frac{1}{2} \#(F, (z_{n_{\pm}}, z_{n_{\pm}}^{-1})) &= \text{even} \geq 0, \quad \#(F, I_j) = \text{even} \geq 2, \\ I_0 &= (z_{-1}, z_1), \quad I_j = (z_j, z_{j+1}), \quad j \in \mathbb{Z}_N \setminus \{-1, n_{n_+}\} = \{n_-, \dots, -2, 1, 2, \dots, n_+ - 1\}. \end{aligned} \quad (1.15)$$

We describe the properties of  $s, w$  and the sequence of zeros  $E_N = (z_j)_{j \in \mathbb{Z}_N} \subset (-1, 1)^N$ .

**Proposition 1.2.** *Let  $q \in \mathfrak{X}_{\nu}^{\tau}$  for some  $\tau, \nu \in \{0, 1\}$ . Then  $(s, E_N) \in \mathcal{S}_{\nu}^{\tau}, w \in \mathcal{W}_{\nu}^{\tau}$  and for each  $j \in \mathbb{Z}_N$  the following identities hold true*

$$\psi^+(z_j) = B(z_j)\psi^-(z_j), \quad B(z_j^{-1})B(z_j) = -1, \quad s(z_j^{-1})s(z_j) = \eta^2(z_j), \quad (1.16)$$

$$m_j^+ = -z_j A'(z_j)B(z_j) = \frac{z_j^2}{\eta^2(z_j)} w'(z_j)s(z_j), \quad (1.17)$$

$$m_j^+ = m_j^- B^2(z_j), \quad (1.18)$$

$$z_j(-1)^j s(z_j^{\pm 1}) > 0, \quad (-1)^j z_j w'(z_j) > 0, \quad (1.19)$$

$$(-1)^{n_{\pm}} w(\pm 1) \geq 0, \quad s(\pm 1) = \pm w(\pm 1). \quad (1.20)$$

Moreover, let  $w(z) = \sum_1^{2p} \check{w}_n z^{n-1}$ ,  $s(z) = \sum_1^{2p} \check{s}_n z^{n-1}$ ,  $\check{w} = (\check{w}_n)_1^{2p}, \check{s} = (\check{s}_n)_1^{2p} \in \mathbb{R}^{2p}$  and let  $Vh = (0, h_1, \dots, h_{2p-1}), h = (h_n)_1^{2p}$ . Then

$$2 + (\check{s}, \check{s}) = (\check{w}, \check{w}), \quad (V^2 \check{s}, \check{s}) = 1 + (V^2 \check{w}, \check{w}) \quad (V^{2k-1} \check{s}, \check{s}) = (V^{2k-1} \check{w}, \check{w}), k = 1, \dots \quad (1.21)$$

Below we will sometimes write  $w(z, q), s(z, q), \dots$ , instead of  $w(z), s(z), \dots$ , when several potentials are being dealt with. For  $q \in \mathfrak{X}_{\nu}^{\tau}$  the iso-resonance set of potentials is given by

$$\text{Iso}(q) = \{r \in \mathfrak{X}_{\nu}^{\tau} : w(\cdot, q) = w(\cdot, r)\}. \quad (1.22)$$

We will describe  $\text{Iso}(q)$ . Assume that we know  $w$  and we need to recover the polynomial  $s$ . Due to Theorem 1.1 the function  $F(z) = w(z)w(z^{-1}) + \eta^2(z) = s(z)s(z^{-1})$  has the zeros  $t_n, t_{n+m} = t_n^{-1}, n = 1, \dots, m$  counted with multiplicity and given by

$$\begin{aligned} 0 < |t_1| \leq \dots \leq |t_m| \leq 1, \quad (t_n)_1^m \subset U = \mathbb{D}_1 \cup \overline{\mathbb{S}}_+^1, \quad \mathbb{S}_+^1 = \mathbb{C}_+ \cap \mathbb{S}^1, \\ \arg t_n \in [0, 2\pi), \quad \text{and} \quad \text{if } |t_n| = |t_k|, \arg t_n \leq \arg t_k \Rightarrow n \leq k. \end{aligned} \quad (1.23)$$

Note that if  $|t_n| = 1$ , then  $\text{Im } t_n \geq 0$ . Hence  $(t_n)_1^m$  is a uniquely defined sequence of all zeros  $\neq 0$  of  $F$  in the set  $U$ . Thus  $t_n, t_n^{-1}, n = 1, \dots, m$  are all zeros of  $F$  and  $t_n$  or  $t_n^{-1}$  is a zero of  $s$ .

If  $q \in \mathfrak{X}_{\nu}^{\tau}$ , then  $s = C_s z^{\nu} \prod_1^m (z - \zeta_n)$ , where each  $\zeta_n \neq 0, n = 1, \dots, m$  and recall that  $\nu \in \{0, 1\}$ . The sequence  $\sigma = (\sigma_n)_0^m$  is defined by

$$\sigma = (\sigma_n)_0^m \in \{\pm 1\}^{m+1}, \quad \sigma_0 = \text{sign } C_s, \quad \text{and} \quad \zeta_n = t_n^{\sigma_n}, \quad n = 1, \dots, m. \quad (1.24)$$

For each  $w \in \mathcal{W}_\tau^\nu$  we define a set  $\Xi_w$  of all possible sequences  $\sigma = (\sigma_n)_0^m$  by

$$\Xi_g = \left\{ \sigma = (\sigma_n)_0^m \subset \{-1, 1\}^{m+1} : (s, E_N) \in \mathcal{S}_\nu^\tau, \text{ where } s = Cz^\nu \prod_1^m (z - t_n^{\sigma_n}), \sigma_0 = \text{sign } C \right\}.$$

In particular we have:

- I) If  $w(1) = w(-1) = 0 = N$ , then  $(\sigma_n)_0^m$  is any sequence from  $\{-1, 1\}^{m+1}$ , under the condition that  $f$  is real.
- II) If  $w(1) \neq 0$  (or  $w(-1) \neq 0$ ), then  $(-1)^{n+s}(1) > 0$  (or  $(-1)^{n-s}(1) < 0$ ) gives  $\sigma_0 = \text{sign } C_s$ .
- III) If  $N \geq 1$ , then condition  $(-1)^j z_j s(z_j) > 0$  for some  $j \in \mathbb{Z}_N$  gives  $\sigma_0 = \text{sign } C_s$ .
- IV) If  $N \geq 2$ , then the function  $s$  has an odd number  $\geq 1$  of zeros on each of the intervals  $(z_{n-}, z_{n-+1}), \dots, (z_{-2}, z_{-1}), (z_{-1}, z_1)$  and  $(z_1, z_2), \dots, (z_{n_+-1}, z_{n_+})$ .

Our goal is to show that the spectral data  $\Xi_w$  give the "proper" parametrization of the set  $\text{Iso}(q)$ . Our main Theorem 1.3 shows that  $\sigma \in \Xi_w$  are **almost free parameters**. Namely, we prove that if the function  $w(z, q)$  is fixed, then each  $\sigma_n$  can be changed in an almost arbitrary way.

**Theorem 1.3.** Let  $\tau, \nu \in \{0, 1\}$  and  $m \geq 3$ .

- i) The mapping  $\mathfrak{J}_R : \mathfrak{X}_\nu^\tau \rightarrow \{(w, \Xi_w), w \in \mathcal{W}_\nu^\tau\}$  given by  $q \rightarrow (w, (\sigma_n)_0^m)$  is one-to-one and onto.
- ii) Let  $q \in \mathfrak{X}_\nu^\tau$ . Then the mapping  $\Psi : \text{Iso}(q) \rightarrow \Xi_w$ , given by  $r \rightarrow \sigma(r)$  (see (1.24)) is a bijection between the set of potentials  $r \in \text{Iso}(q)$  and the set of sequences  $\sigma(r) \in \Xi_w, w = w(\cdot, q)$ .

A great number of papers are devoted to the inverse problem for the Schrödinger operator, (see a book [M], papers [Fa], [DT], [Me] and ref. therein).

A lot of papers are devoted to the resonances for the 1D Schrödinger operator, see [F], [K1], [K2], [K3], [S], [Z], [Z1]. We recall that Zworski [Z] obtained the first results about the distribution of resonances for the Schrödinger operator with compactly supported potentials on the real line. Korotyaev obtained the characterization (plus uniqueness and recovering) of  $S$ -matrix for the Schrödinger operator with a compactly supported potential on the real line [K1] and on the half-line [K2]. In [K3] for the Schrödinger operator on the half line the stability result was given: if  $\varkappa^0 = \{\varkappa^0\}_1^\infty$  is a sequence of zeros (eigenvalues and resonances) of the Jost function for some real compactly supported potential  $q_0$  and  $\varkappa - \varkappa^0 \in \ell_\varepsilon^2$  for some  $\varepsilon > 1$ , then  $\varkappa$  is the sequence of zeros of the Jost function for some unique real compactly supported potential.

There are a lot of papers and books devoted to the scattering for Jacobi operators, see [C1], [C2], [CC], [CK], [G1], [G2], [NMPZ], [Te], [T1], [T2]. In the case of Jacobi operators also there are papers about the inverse resonance problem, see [BNW], [MW], [DS1], [DS2]. In particular, some progress was made by Damanik and Simon in [DS1], [DS2], where they described the  $S$ -matrix for Jacobi operators on the half-lattice both for finite-support and exponentially decay perturbations.

## 2 Proof of main Theorems

We recall well-known facts from [Te].

**Lemma 2.1.** *Let  $q \in \ell_{1,2p}^2$ . Then each function  $\psi_{p-n}^+(z), n = 1, 2, \dots, p$  is a real polynomial and satisfies*

$$\psi_p^+(z) = \frac{z^p}{a_p}, \quad \psi_{p-n}^+(z) = \frac{z^{p+n}}{\eta_n} \left( c_p - \frac{c_p \beta_{p-n+1} + b_p a_p^2}{z} + \frac{O(1)}{z^2} \right), \quad c_n = 1 - a_n^2, \quad (2.1)$$

$$\psi_2^+(z) = \frac{z^{2p-2}}{\eta_2} \left( c_p - \frac{c_p \beta_3 + b_p a_p^2}{z} + \frac{O(1)}{z^2} \right), \quad (2.2)$$

$$\psi_1^+(z) = \frac{z^{2p-1}}{\eta_1} \left( c_p - \frac{c_p \beta_2 + b_p a_p^2}{z} + \frac{O(1)}{z^2} \right) \quad (2.3)$$

as  $z \rightarrow \infty$ , where  $\beta_n = b_n + b_{n+1} + \dots + b_p$  and  $\eta_n = a_n a_{n+1} \cdots a_p$ . Moreover,

$$\psi_{p-n}^+(z) = \frac{z^{p-n}}{\eta_{p-n}} (1 - z \beta_{p-n+1} + O(z^2)), \quad \psi_1^+(z) = z \frac{1 - z \beta_2 + O(z^2)}{\eta_1} \quad \text{as } z \rightarrow 0, \quad (2.4)$$

$$\psi_1^-(z) = z^{-1}, \quad \psi_2^-(z) = \frac{1 - z b_1}{a_1 z^2}. \quad (2.5)$$

**Lemma 2.2.** *Let  $q \in \ell_{1,2p}^2$ . Then*

$$w(z) = -(b_1 - z^{-1})\psi_1^+(z) - a_1\psi_2^+(z), \quad z \neq 0, \quad (2.6)$$

$$s(z) = (1 - b_1 z^{-1})\psi_1^+(z) - z^{-1}a_1\psi_2^+(z), \quad (2.7)$$

$$w(z) = \frac{z^{2p-1}}{\eta_1} \left( -b_1 c_p + \frac{c_p(c_1 + \beta_2 b_1) + b_1 b_p a_p^2}{z} + \frac{O(1)}{z^2} \right) \quad \text{as } z \rightarrow \infty$$

$$w(z) = \frac{1 - z \beta_1 + O(z^2)}{\eta_1} \quad \text{as } z \rightarrow 0, \quad (2.8)$$

$$s(z) = \frac{z^{2p-1}}{\eta_1} \left( c_p - \frac{c_p \beta_1 + b_p a_p^2}{z} + \frac{O(1)}{z^2} \right) \quad \text{as } z \rightarrow \infty$$

$$s(z) = \frac{1}{\eta_1} \left( -b_1 + z(c_1 + b_1 \beta_2) + O(z^2) \right) \quad \text{as } z \rightarrow 0. \quad (2.9)$$

Furthermore, if  $b_1 = c_p = 0$ , then

$$w(z) = -\frac{z^{2p-3}}{\eta_1} (c_1 b_p + O(1/z)) \quad \text{as } z \rightarrow \infty. \quad (2.10)$$

**Proof.** Using (1.6) and Lemma 2.1, we obtain (2.6), (2.7). Then asymptotics from Lemma 2.1 imply (2.8), (2.9). If  $b_1 = c_p = 0$ , then (2.2), (2.3) imply

$$w = -a_1\psi_2^+ + \frac{\psi_1^+}{z} = \frac{a_1^2 z^{2p-3}}{\eta_1}(b_p + O(z^{-1})) - \frac{z^{2p-3}}{\eta_1}(b_p + O(z^{-1})) = -\frac{z^{2p-3}}{\eta_1}(c_1 b_p + O(z^{-1})).$$

■

**Proof Proposition 1.2.** Identities (1.16) follow from (1.5), (1.8). Recall the following identity

$$A'(z_j) = -\frac{1}{z_j} \sum_{n \in \mathbb{Z}} \psi_n^+(z_j) \psi_n^-(z_j), \quad j \in \mathbb{Z}_N, \quad (2.11)$$

see (10.34) in [Te]. Then using (1.16) we obtain (1.17). Similar arguments give (1.18).

Using  $w(0) > 0$  and (1.17), we obtain  $w'(z_1) < 0$  and  $w'(z_2) > 0, \dots$ . Moreover, due to (1.16), (1.17) we have (1.19). Identity (2.6), (2.7) give (1.20).

Substituting  $w = \sum_1^{2p} \tilde{w}_n z^{n-1}$  and  $s = \sum_1^{2p} \tilde{s}_n z^{n-1}$  into (1.8) we obtain (1.21). ■

We need some results about the inverse problems from [Te] for  $q \in \ell_1^1 = \{h = (h_n)_{n \in \mathbb{Z}} : \sum(1 + |n|)|h_n| < \infty\}$ . Define the Marchenko operator  $\mathfrak{F}_n : \ell^2(\mathbb{Z}_+) \rightarrow \ell^2(\mathbb{Z}_+)$  by

$$(\mathfrak{F}_n f)_k = \sum_{m \geq 0} F(2n + m + k) f_m, \quad f = (f_n)_0^\infty \in \ell^2(\mathbb{Z}_+), \quad \mathbb{Z}_+ = \mathbb{Z} \cap [0, \infty), \quad (2.12)$$

$n, k \geq 0$  where

$$F(n) = \hat{R}_n^+ + \sum_{j=1}^N \frac{z_j^n}{m_j}, \quad \hat{R}_n^\pm = \frac{1}{2\pi i} \int_{|z|=1} R_\pm(z) z^{n-1} dz, \quad n \in \mathbb{Z}, \quad (2.13)$$

and  $\hat{R}_n^\pm$  are the Fourier coefficients of  $R_\pm$ . The Marchenko equation is given by

$$(I + \mathfrak{F}_n) \mathcal{K}_n(\cdot) = \eta_n^2 e_0, \quad \text{where } \mathcal{K}_n = (\mathcal{K}_n(m))_0^\infty, \quad e_n = (\delta_{n,k})_0^\infty \in \ell^2(\mathbb{Z}_+), \quad (2.14)$$

i.e.,

$$\mathcal{K}_n(k) + F(2n + k) + \sum_{m \geq 1} F(2n + k + m) \mathcal{K}_n(m) = \eta_n^2 \delta_{0,k}. \quad (2.15)$$

For each  $n \in \mathbb{Z}$  these equations have unique "decreasing" solutions  $(\mathcal{K}_n(k))_{k=0}^\infty$ . The sequences  $a_n, b_n, n \in \mathbb{Z}$  have the forms

$$a_n^2 = \frac{\Psi_n^0}{\Psi_{n+1}^0}, \quad b_n = \frac{\Psi_n^1}{\Psi_n^0} - \frac{\Psi_{n-1}^1}{\Psi_{n-1}^0}, \quad \Psi_n^k = \langle e_k, (I + \mathfrak{F}_n) e_0 \rangle, \quad k = 0, 1, \quad n \in \mathbb{Z}, \quad (2.16)$$

where  $\langle \cdot, \cdot \rangle$  is the scalar product in  $\ell^2(\mathbb{Z}_+)$ . Recall the inverse spectral theorem from [Te].

**Theorem 2.3.** *The Faddeev mapping*

$$\ell_1^1(\mathbb{Z}) \oplus \ell_1^1(\mathbb{Z}) \rightarrow \mathfrak{S} = \{R_+, (z_j, m_j^+, j \in \mathbb{Z}_N) \in (-1, 1)^N \times \mathbb{R}_+^N, z_j \neq 0, N \geq 0\} \quad (2.17)$$

given by  $(a_n - 1, b_n)_{n \in \mathbb{Z}} \rightarrow \{R_+(z), z \in \mathbb{S}^1, (z_j, m_j^+, j \in \mathbb{Z}_N)\}$  is one-to-one and onto, where the function  $R_+$ , the eigenvalues  $z_j$  and the norming constants  $m_j^+, j \in \mathbb{Z}_N$  satisfy the conditions

1)  $R_+(z) = \overline{R_+}(\overline{z}) = -\frac{s(\overline{z})}{zw(z)}, z \in \mathbb{S}_0 = \mathbb{S}^1 \setminus \{\pm 1\}$ , and the function  $R_+(z)$  is continuous in  $z \in \mathbb{S}_0$  and satisfies

$$C|1 - z^2|^2 + |R_+(z)|^2 \leq 1, \quad \text{all } z \in \mathbb{S}_0, \quad \text{for some } C > 0. \quad (2.18)$$

2) The eigenvalues  $z_j, j \in \mathbb{Z}_N$  are distinct and  $m_j^- m_j^+ = w'(z_j)^2$ .

3) The sequences  $\hat{R}^\pm = (\hat{R}_n^\pm)_1^\infty$  defined in (2.13) with  $R_-(z) = -R_+(\overline{z}) \frac{A(\overline{z})}{A(z)}$  satisfy

$$\sum_{n \geq 1} n |\hat{R}_n^\pm - \hat{R}_{n+2}^\pm| < \infty. \quad (2.19)$$

We are ready to prove the first result.

**Proof of Theorem 1.1.** We consider the case  $q \in \mathfrak{X}_0^0$  and  $m = 2p - 1$ , the proof of other cases is similar. If  $q \in \mathfrak{X}_0^0$ , then Lemma 1.2 gives that  $(s, E_N) \in \mathcal{S}_0^0$ , which yields a mapping  $q \rightarrow (s, E_N)$  from  $\mathfrak{X}_0^0$  into  $\mathcal{S}_0^0$ .

We will show uniqueness. Let  $q \in \mathfrak{X}_0^0$ . Then Lemma 1.2 gives that  $(s, E_N) \in \mathcal{S}_0^0$ , and Theorem 3.2 gives a unique  $w$ . Moreover, Proposition 1.2 yields the norming constants  $m_j^\pm, j \in \mathbb{Z}_N$ . These data determine the compactly supported potential uniquely by Theorem 2.3. Then we deduce that the mapping  $q \rightarrow (s, E_N)$  is an injection.

We will show surjection of the mapping  $q \rightarrow (s, E_N)$ . If  $(s, E_N) \in \mathcal{S}_0^0$ , then Theorem 3.2 gives unique  $w \in \mathcal{W}_0^0$  and we have  $R_+ = -\frac{s(z^{-1})}{zw(z)}$ . If  $n \geq 2p + 1$ , then we have

$$\hat{R}_n^+ = \frac{1}{2\pi i} \int_{|z|=1} R_+(z) z^{n-1} dz = -\frac{1}{2\pi i} \int_{|z|=1} \frac{z^{n-2} \tilde{s}(z)}{w(z)} dz = -\sum_{j=1}^N \text{Res} \frac{z^{n-2} \tilde{s}(z)}{w(z)} \Big|_{z_j} = -\sum_{j=1}^N \frac{z_j^n}{m_j},$$

since  $n - 2 = (2p - 1) + (n - 2p - 1)$  and the function  $z^{2p-1} \tilde{s}$  is a polynomial. Thus

$$F(n) = 0 \quad \text{if} \quad n \geq 2p + 1, \quad (2.20)$$

and using (2.16) we obtain

$$\Psi_n^0 = 1 + F(2n), \quad \text{and} \quad a_n^2 = \frac{1 + F(2n)}{1 + F(2n + 2)} \quad (2.21)$$

thus  $a_n^2 = 1$  if  $n > p$ . Similar arguments yield  $b_n = 0$  if  $n > p$ . Then we deduce that

$$a_n = 1, \quad b_n = 0 \quad \text{if} \quad n \geq p + 1.$$

Moreover, similar arguments yield  $b_n = 0, a_n = 1$  if  $n \leq 0$ . Then the asymptotics from Lemma 2.2 give that  $q \in \mathfrak{X}_0^0$ , which yields surjection. ■

**Proof of Theorem 1.3.** i) We consider the case  $q \in \mathfrak{X}_0^0$ , the proof of other cases is similar. Let  $q \in \mathfrak{X}_0^0$ . Then Proposition 1.2 yield  $w(\cdot, q) \in \mathcal{W}_0^0$  and we have the mapping  $q \rightarrow (w, \sigma)$ , where the sequence  $\sigma \in \Xi_w$  is given by (1.24), since  $(s, E_N) \in \mathcal{S}_0^0$ .



By Theorem 3.3, for each  $(w, \sigma) \in (w, \Xi_w)$  there exists a unique  $(s, E_N) \in \mathcal{S}_0^0$ , then due to Theorem 1.1 the mapping  $q \rightarrow (w, \sigma)$  is an injection.

We will prove that  $\mathfrak{J}_R$  is a surjection. Let  $w \in \mathcal{W}_0^0$  have zeros  $E_N = (z_j)_{j \in \mathbb{Z}_N}$  from  $(-1, 1)$ ,  $N \geq 0$  and let a sequence  $\sigma = (\sigma_n)_0^m \in \Xi_w$  be defined by (1.24). By Theorem 3.2, there exists a unique  $(s, E_N) \in \mathcal{S}_0^0$  such that (1.12) with  $E_N$  hold true. Then  $(s, E_N) \in \mathcal{S}_0^0$  and, by Theorem 1.1, there exists a unique potential  $q \in \mathfrak{X}_0^0$  with the scattering function  $s(\cdot)$ . Thus the mapping  $\mathfrak{J}_{res} : \mathfrak{X}_\nu^\tau \rightarrow ((w, \sigma) \in \mathcal{W}_\nu^\tau \times \Xi_w)$  given by  $q \rightarrow (w, (\sigma_n)_0^m)$  is one-to-one and onto, where  $m = 2p - 1$ .

ii) Furthermore, using similar arguments we deduce that if we fix  $q \in \mathfrak{X}_\nu^\tau$ , then the mapping  $\Psi : r \rightarrow \sigma(s(r))$  is a bijection between the iso-resonance set of potentials  $\text{Iso}(q)$  and the set of sequences  $\Xi_w, w = w(\cdot, q)$ . ■

### 3 Properties of polynomials $s, w$

In this section we will get the needed results about polynomials  $s, w$ . In order to solve the inverse problems  $(a, b) \rightarrow (\text{spectral data})$  we need the following results.

**Lemma 3.1.** *Let  $g = C_g \prod_1^m (z - \rho_n)$  and  $f = C_f \prod_1^m (z - \zeta_n)$  satisfy*

$$g(z)g(z^{-1}) + \eta^2 = f(z)f(z^{-1}), \quad \eta = z - z^{-1}, \text{ all } z \neq 0, \quad (3.1)$$

*for some  $m \geq 3$  and  $g(0) \neq 0, f(0) \neq 0$ . Then*

$$\prod_1^m (\lambda - \mu_n) = \frac{\lambda^2 - 4}{C} + \prod_1^m (\lambda - \lambda_n), \quad \mu_n = \zeta_n + \frac{1}{\zeta_n}, \quad \lambda_n = \rho_n + \frac{1}{\rho_n}, \quad (3.2)$$

$$C = C_s^2 C_\zeta = C_w^2 C_\rho, \quad \frac{C_\rho}{C_\zeta} > 0, \quad C_\rho = \prod_1^m (-\rho_n), \quad C_\zeta = \prod_1^m (-\zeta_n), \quad (3.3)$$

$$\sum_1^m \mu_n = \sum_1^m \lambda_n, \quad \dots, \quad \prod_1^m (-\mu_n) + \frac{4}{C} = \prod_1^m (-\lambda_n). \quad (3.4)$$

**Proof.** Using  $f = C_f \prod_1^m (z - \zeta_n)$  and  $g = C_g \prod_1^m (z - \rho_n)$ , we obtain

$$C_s^2 \prod_1^m (z - \zeta_n)(z^{-1} - \zeta_n) = \eta^2 + C_w^2 \prod_1^m (z - \rho_n)(z^{-1} - \rho_n).$$

Thus the identities  $-\zeta_n(\lambda - \mu_n) = (z - \zeta_n)(z^{-1} - \zeta_n)$  and  $-\rho_n(\lambda - \lambda_n) = (z - \rho_n)(z^{-1} - \rho_n)$ , give

$$C_s^2 C_\zeta \prod_1^m (\lambda - \mu_n) = \lambda^2 - 4 + C_w^2 C_\rho \prod_1^m (\lambda - \lambda_n),$$

and  $C_s^2 C_\zeta = C_w^2 C_\rho = C$ , which yields  $\frac{C_\zeta}{C_\rho} = \frac{C_w^2}{C_s^2} > 0$ . ■

Assume that we know only the polynomial  $f$  in the equation (3.1) and we have to determine  $g$ . In order to do this we have to solve the equation (3.1) in some class of polynomials. The following Theorem will be used to determine the polynomial  $w$  if we know  $s$ .

**Theorem 3.2.** *i) Let functions  $f, g$  be analytic in  $\mathbb{C} \setminus \{0\}$  and satisfy (3.1) and be real on  $\mathbb{R} \setminus \{0\}$ . Then  $f^2(\pm 1) = g^2(\pm 1)$ . Moreover, if  $f(\pm 1) = 0$  then  $(f')^2(\pm 1) = 8 + (g')^2(\pm 1) \geq 8$ .  
ii) Let  $(f, K_f^N) \in \mathcal{S}_\nu^\tau$ , for some  $(\nu, \tau) \in \{0, 1\}$ ,  $N \geq 0, m \geq 3$ . Then there exists a unique polynomial  $g \in \mathcal{W}_\nu^\tau$  satisfying (3.1).*

**Remark.** It is possible that the function  $f(z)f(z^{-1}) - \eta^2(z)$  has more zeros on  $(-1, 1) \setminus \{0\}$ , and it is important that we make a special choice which, however, has to satisfy condition (1.15).

**Proof.** The statement i) is very simple and differentiating (3.1) we obtain  $(f')^2(\pm 1) = 8 + (f'')^2(\pm 1)$  at  $z = \pm 1$ .

ii) Recall that  $f = z^\nu C \prod_1^m (z - \zeta_n)$ , where

$$C \in \mathbb{R} \setminus \{0\}, \quad 0 < |\zeta_1| \leq |\zeta_2| \leq \dots \leq |\zeta_m|, \quad m = 2p - 1 - \tau - \nu,$$

and  $K_f^N = (z_j)_{j \in \mathbb{Z}_N}$  is some sequence of zeros of the function  $f(z)f(z^{-1}) - \eta^2(z)$  such that

$$-1 < z_{n_-} < \dots < z_{-1} < 0 < z_1 < \dots < z_{n_+} < 1, \quad (-1)^{n_\pm} f(\pm 1) \geq 0, \quad z_j (-1)^j f(z_j^{\pm 1}) > 0, \quad (3.5)$$

$j \in \mathbb{Z}_N$ . Then

$$f(z)f(z^{-1}) = C^2 \prod_1^m (z - \zeta_n)(z^{-1} - \zeta_n) = \frac{C_0}{z^m} \prod_1^m (z - \zeta_n)(z - \zeta_n^{-1}),$$

where  $C_0 = C^2 C_\zeta$ ,  $C_\zeta = \prod_1^m (-\zeta_n)$ . Then  $G(z) = f(z)f(z^{-1}) - \eta^2(z)$  satisfies

$$G(z) = \begin{cases} C_0 z^m (1 + O(z^{-1})) & \text{as } z \rightarrow \infty \\ C_0 z^{-m} (1 + O(z)) & \text{as } z \rightarrow 0 \end{cases}, \quad (3.6)$$

and thus

$$G(z) = -\eta^2(z) + \frac{C_0}{z^m} \prod_1^m (z - \zeta_n)(z - \zeta_n^{-1}) = \frac{C_0}{z^m} \prod_1^m (z - \rho_n)(z - \rho_n^{-1}), \quad (3.7)$$

where  $\rho_n \neq 0, \rho_n^{-1}$  are the zeros of  $G$  counted with multiplicity and satisfying

$$0 < |\rho_1| \leq |\rho_2| \leq \dots \leq |\rho_N| < 1 \leq |\rho_{N+1}| \leq \dots \leq |\rho_m|, \\ \text{the set } \{\rho_1, \rho_2, \dots, \rho_N\} = \{z_j, j \in \mathbb{Z}_N\} \subset (-1, 1) \setminus \{0\}, \quad (3.8)$$

and Conditions i)-iv) in Definition W, since  $f$  satisfies Definition S and  $G(z) = G(1/z)$  for all  $z \neq 0$ . Moreover, using  $g_0(z) = \prod_1^m (z - \rho_n)$  we have

$$G(z) = C g_0(z) g_0(z^{-1}), \quad \text{where} \quad C = \frac{C_0}{g_0(0)}, \quad (3.9)$$

Note that Lemma 3.1 gives  $C > 0$ , then  $g = C_* g_0$  and  $g(0) > 0$ , where  $C_*$  satisfies  $C_* g_0(0) > 0, C_*^2 = C > 0$ . By the construction of  $g$ , this function is unique. ■

In order to solve the inverse problems  $\mathfrak{X}_\nu^\tau \rightarrow \mathcal{W}_\nu^\tau$  we need the following results. Assume that we have  $(w, \sigma)$ , where the function  $w \in \mathcal{W}_\nu^\tau$  and the sequence  $\sigma \in \Xi_w$ , then we have to determine  $s$  uniquely. In order to do this we have to solve the equation (3.1) in class of polynomials  $s \in \mathcal{S}_\nu^\tau$ . The sequence  $\sigma$  will give uniqueness. The following Theorem will be used to determine the function  $s$  if we know  $(w, \sigma)$ .

**Theorem 3.3.** *Let  $g \in \mathcal{W}_\nu^\tau$ , for some  $\nu, \tau \in \{0, 1\}, m \geq 3$ . Then for each  $\sigma \in \Xi_g$  defined in (1.24) there exists a unique  $(f, K_f^N) \in \mathcal{S}_\nu^\tau$  satisfying (3.1).*

**Proof.** Recall that  $g = C_g \prod_1^m (z - \rho_n)$ ,  $m = 2p - 1 - \tau - \nu$  for some  $C_g \in \mathbb{R} \setminus \{0\}$  and  $\rho_n \in \mathbb{C} \setminus \{0\}$  such that:

- i)  $g$  is real on  $\mathbb{R}$  and if  $+1$  and  $-1$  are zeros, they are simple, and  $g(0) > 0$ ,
- ii)  $|g(z)| \geq |\eta(z)|$  for any  $|z| = 1$ , where  $\eta = z - \frac{1}{z}$ ,
- iii)  $g$  has only simple zeros  $z_{n_-}, \dots, z_{-1}, z_1, \dots, z_{n_+}$  in  $\mathbb{D}_1$  for some  $\pm n_\pm \geq 0$  such that  $-1 < z_{n_-} < \dots < z_{-1} < 0 < z_1 < \dots < z_{n_+} < 1$ .

Then we obtain

$$g(z)g(z^{-1}) = C_g^2 \prod_1^m (z - \rho_n)(z^{-1} - \rho_n) = \frac{C_0}{z^m} \prod_1^m (z - \rho_n)(z - \rho_n^{-1}),$$

where  $C_0 = C_g^2 \prod_1^m (-\rho_n)$ . Then  $F(z) = \eta^2(z) + g(z)g(z^{-1})$  satisfies

$$F(z) = \begin{cases} C_0 z^m (1 + O(z^{-1})) & \text{as } z \rightarrow \infty \\ C_0 z^{-m} (1 + O(z)) & \text{as } z \rightarrow 0 \end{cases}, \quad (3.10)$$

and thus

$$F(z) = \frac{C_0}{z^m} \prod_1^{2m} (z - t_n) = \frac{C_0}{z^m} \prod_1^m (z - t_n)(z - t_n^{-1}), \quad (3.11)$$

where  $t_n \neq 0$  are the zeros of  $F$  counted with multiplicity and satisfying

$$0 < |t_1| \leq |t_2| \leq \dots \leq |t_m|, \quad t_{n+m} = 1/t_n, \text{ all } n = 1, 2, \dots, m, \quad \arg t_n \in [0, 2\pi),$$

where if  $|t_n| = |t_k|$ ,  $\arg t_n \leq \arg t_k \Rightarrow n \leq k$  and if  $|t_n| = 1$ , then  $\text{Im } t_n \geq 0$ , since  $F(z) = F(1/z)$  for all  $z \neq 0$ . Moreover, we have

$$F(z) = \frac{C_0}{z^m} \prod_1^m (z - \zeta_n)(z - \zeta_n^{-1}) = \frac{C_0}{f_0(0)} f_0(z) f_0(z^{-1}), \quad f_0(z) = \prod_1^m (z - \zeta_n), \quad (3.12)$$

where  $\zeta_n = t_n^{\sigma_n}$ ,  $(\sigma_n)_1^m \in \Xi_g$ . Note that Lemma 3.1 gives  $E = \frac{C_0}{f_0(0)} > 0$ , then  $f = C_* f_0$  and  $g(0) > 0$ , where  $C_* = \sqrt{E}$ . We need to choose the sign of  $C_*$ . By the construction of  $g$ , this function is unique. ■

**Example p=1.** In this case we have

$$w = \frac{1}{a_1} (1 - b_1 z - a_1^2 z^2), \quad s = \frac{1}{a_1} (-b_1 + c_1 z). \quad (3.13)$$

1) If we know  $s = s_0 + 2s_1z$ , then

$$a_1 = -s_1 + \sqrt{s_1^2 + 1}, \quad b_1 = -\frac{s_0}{s_1}. \quad (3.14)$$

2) If we know  $w$ , then

$$-a_1w = z^2 + \frac{b_1}{a_1^2}z - \frac{1}{a_1^2}, \quad \rho_1 = \frac{-b_1 - \sqrt{b_1^2 + 4a_1^2}}{4a_1^2} < 0, \quad \rho_2 = \frac{-b_1 + \sqrt{b_1^2 + 4a_1^2}}{4a_1^2} > 0, \quad (3.15)$$

where  $\rho_1, \rho_2 \in \mathbb{R}$ . Then

$$\rho_1\rho_2 = -\frac{1}{a_1^2}, \quad b_1 = \frac{\rho_1 + \rho_2}{\rho_1\rho_2}.$$

**Example p=2.** In this specific case we have

$$\psi_2^+ = \frac{z^2}{a_2}, \quad \psi_1^+ = \frac{c_2z^3 - b_2z^2 + z}{a_1a_2}, \quad (3.16)$$

and

$$w = \frac{1}{a_1a_2} \left( -z^3c_2b_1 + z^2(c_2 - a_1^2 + b_1b_2) - z(b_1 + b_2) + 1 \right), \quad (3.17)$$

$$s = \frac{1}{a_1a_2} \left( z^3c_2 - z^2(b_2 + b_1c_2) + z(c_1 + b_1b_2) - b_1 \right). \quad (3.18)$$

We obtain 4 cases:

1) The case  $a_2 \neq 1, b_1 \neq 0$  have been considered since  $m = 3$ .

2) Let  $a_2 = 1, b_2 \neq 1, b_1 = 0, a_1 \neq 1$ . Then it is similar to the case  $p = 1$  and we have:

$$w = \frac{-a_1^2z^2 - zb_2 + 1}{a_1}, \quad s = z \frac{-zb_2 + c_1}{a_1}. \quad (3.19)$$

3) If  $a_2 = 1, b_2 \neq 0, b_1 \neq 0$ , then

$$w = \frac{z^2(b_1b_2 - a_1^2) - z(b_1 + b_2) + 1}{a_1}, \quad s = \frac{-z^2b_2 + z(c_1 + b_1b_2) - b_1}{a_1}. \quad (3.20)$$

4) If  $a_2 \neq 1, b_1 = 0, a_1 \neq 1$ , then

$$w = \frac{z^2(c_2 - a_1^2) - zb_2 + 1}{a_1a_2}, \quad s = z \frac{z^2c_2 - zb_2 + c_1}{a_1a_2}. \quad (3.21)$$

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